

IMBEDDINGS INTO GROUPS OF INTERMEDIATE GROWTH

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ABSTRACT. Every countable group that does not contain a finitely generated subgroup of exponential growth imbeds in a finitely generated group of subexponential growth.

This produces in particular the first examples of groups of subexponential growth containing the additive group of the rationals.

This also produces groups of subexponential growth and arbitrarily large distortion in uniformly convex Banach (e.g. Hilbert) spaces.

1. INTRODUCTION

A classical result by Higman, Neumann and Neumann [12] states that every countable group imbeds in a finitely generated group. It was then shown that many properties of the group can be inherited by the imbedding: in particular, solvability (Neumann-Neumann [16]), torsion (Phillips [18]), residual finiteness (Wilson [19]), and amenability (Olshansky-Osin [17]).

Seen the other way round, these results show that there is little restriction, apart from being countable, on the subgroups of a finitely generated group.

A finitely generated group G has *polynomial growth* if there is a polynomial function $p(n)$ bounding from above the number of group elements that are products of at most n generators, has *subexponential growth* if $p(n)$ may be chosen subexponential in n , and has *intermediate growth* if G has subexponential but not polynomial growth.

By a theorem of Gromov [10], groups of polynomial growth are virtually nilpotent, so all its subgroups are finitely generated (see e.g. [14, Corollary 9.10]). On the other hand, there are groups of intermediate growth such as the “first Grigorchuk group” [8] with infinitely generated subgroups. We are therefore led to ask which groups may appear as subgroups of a group of subexponential growth.

1.1. Main result. Let us say that a group has *locally subexponential growth* if all of its finitely generated subgroups have subexponential growth. Clearly, if G has subexponential growth then all its subgroups have locally subexponential growth. Our main result shows that this is the only restriction:

Theorem A. *Let B be a countable group of locally subexponential growth. Then there exists a finitely generated group of subexponential growth in which B imbeds.*

In contrast, there exist nilpotent (and even abelian) countable groups that do not imbed into finitely generated nilpotent groups. Gromov’s theorem mentioned above has the consequence that there exist countable groups of locally polynomial growth that do not imbed in groups of polynomial growth. Mann noted in [14, Corollary 9.11] that torsion-free groups locally of polynomial growth of bounded degree are also virtually nilpotent.

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It is a tantalizing open question to understand which properties are shared by groups of intermediate growth and by nilpotent and virtually nilpotent groups. It is clear that a group of intermediate growth cannot contain non-abelian free subgroups or even free subsemigroups. Groups of intermediate growth were constructed by Grigorchuk in [8], and his first example, known as the “first Grigorchuk group”, admits a pair of dilating endomorphisms with commuting images. This property can be viewed as a higher dimensional analogue of groups with dilation; and any group admitting a dilation has polynomial growth.

We may also ask which groups may appear as subgroups of a specific group of intermediate growth such as the first Grigorchuk group G_{012} . For example, G_{012} is known to contain every finite 2-group, and all its subgroups are countable, residually-2 and have locally smaller growth.

There are other restrictions, apart from these obvious ones, for a countable group to be imbedded as a subgroup of a generalised Grigorchuk group. For example, only a finite number of primes appears as exponents in a Grigorchuk group. Extensions of Grigorchuk groups constructed by the authors in [4] admit a larger class of possible subgroups, but similar restrictions appear nevertheless. In particular, Theorem A gives the first groups of subexponential growth containing \mathbb{Q} .

1.2. Distortion. Consider an 1-Lipschitz map $\Phi : (\mathcal{X}, d) \rightarrow (\mathcal{Y}, d)$ between metric spaces. Its *distortion* is the function

$$\rho_\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad \rho_\Phi(t) = \inf_{d(x, x') \geq t} d(\Phi(x), \Phi(x')).$$

It is the largest increasing function $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\rho(d(x, x')) \leq d(\Phi(x), \Phi(x')) \leq d(x, x') \text{ for all } x \neq x' \in \mathcal{X}.$$

We say that Φ has distortion *better than* ρ if $\rho_\Phi(t) > \rho(t)$ for all $t \in \mathbb{R}_+$ large enough, *worse than* ρ if $\rho_\Phi(t) < \rho(t)$ for all $t \in \mathbb{R}_+$ large enough, and that Φ is a *uniform imbedding* if its distortion is unbounded. See [11, §7.E] for original motivations.

As a consequence of our construction, we construct for every unbounded increasing function $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a group W of subexponential growth, such that every imbedding of W in Hilbert space (or, more generally, uniformly convex Banach spaces of given uniformity modulus) has distortion worse than ρ .

Arzhantseva, Druţu and Sapir construct in [2], for every unbounded increasing function ρ , a group which imbeds uniformly into Hilbert space, and such that all of its imbeddings into Hilbert space have distortion worse than ρ . Olshansky and Osin construct, moreover, amenable groups with this property.

These examples all have exponential growth. In contrast, the main point of our construction is to produce such groups having subexponential growth. These are in particular the first known examples of groups whose simple random walks have trivial Poisson boundary and with arbitrarily bad distortion in every imbedding into Hilbert space. It follows from [15, Theorem 1.1] by Naor and Peres that, if an amenable group G admits an imbedding with distortion better than $n^{1/2-\epsilon}$ for some $\epsilon > 0$, then every simple random walk on G has trivial Poisson boundary. Our result shows that groups with trivial Poisson boundary for every simple random walk on G may have arbitrarily bad distortion in every imbedding in Hilbert space.

Note that groups of subexponential growth are amenable, and therefore imbed uniformly into Hilbert space. In fact, every amenable group G admits imbeddings better than an unbounded function ρ which can be chosen in terms of the Følner function of G .

It was already known that groups of subexponential growth can have arbitrarily large Følner function [7]. Our result is a strengthening of this fact.

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2. SKETCH OF THE PROOFS

The original imbedding result by Higman-Neumann-Neumann [12] mentioned in the introduction proceeds by a sequence of “HNN extensions”. We recall the later construction by Neumann-Neumann, which uses wreath products rather than HNN extensions. The *unrestricted* wreath products of two groups H, G is the group $H \wr G = H^G \rtimes G$, the split extension of the set of maps $G \rightarrow H$ by G . The Neumann-Neumann construction proceeds in two steps:

(i) starting with a countable group B , one imbeds it into a countable subgroup G of the unrestricted wreath product $B \wr \mathbb{Z}$ in such a way that B is imbedded into the commutator group $[G, G]$. The group G is generated by \mathbb{Z} and, for all $b \in B$, the function $f_b: \mathbb{Z} \rightarrow B$ defined by $f_b(m) = b^m$. Denoting by t the generator of \mathbb{Z} , we see that $[t, f_b]$ is the constant function b ; so B is in fact imbedded in $[t, G]$.

(ii) starting with a countable group G , one imbeds the commutator subgroup $[G, G]$ into a two-generated subgroup W of the unrestricted wreath product of $G \wr \mathbb{Z}$. More generally, one constructs imbeddings into $G \wr P$ for a finitely generated group P . Denoting a generating set of G by $\{b_1, b_2, \dots\}$, the group W is generated by P and $f: P \rightarrow G$ with $f(x_i) = b_i$ along a sparse-enough sequence $(x_i)_{i \geq 1}$ of elements of P . In fact, since it suffices in (i) to imbed $[t, G]$ in W , one sets $f(1) = t$ and the exact requirement on the sequence (x_i) is: $x_i \neq 1$ for all i ; all x_i are distinct; and $x_i x_j \notin \{1, x_k\}$ for all $i, j, k \in \mathbb{N}$. One then sees that $[f, f^{x_i^{-1}}]$ is a function supported only at 1, with value $[t, b_i]$ there. This is the imbedding of $[t, G]$.

The combination of both steps imbeds B into the finitely generated group W . If B is solvable, then so are $B \wr \mathbb{Z}$ and G ; and similarly, if G is solvable, then so are $G \wr \mathbb{Z}$ and W .

This construction may be applied to an arbitrary countable group B , but some properties of B , such as amenability, may be lost along the way. Olshansky-Osin introduce in [17] the following slightly stronger condition on $(x_i)_{i \geq 1}$: by definition, a *parallelogram* in a sequence $(x_i)_{i \geq 1}$ is a quadruple of elements $p_1 \neq p_2 \neq p_3 \neq p_4 \neq p_1$, each belonging to $\{x_i\}$, such that $p_1 p_2^{-1} p_3 p_4^{-1} = 1$. A sequence is *parallelogram-free* if it contains no parallelogram. They show that, if (x_i) is parallelogram-free, then the group W is obtained from G and P by extensions, subgroups, quotients and directed limits, so in particular is amenable as soon as G and P are amenable.

They also modify slightly step (i), by defining rather $f_b(m) = b$ for $m \geq 0$ and $f_b(m) = 1$ for $m < 0$; then $[t, f_b]$ is the function supported at 0 with value b there, and the group $G = \langle t, f_b: b \in B \rangle$ is also obtained from B and \mathbb{Z} by elementary operations, so is amenable as soon as B is amenable.

Note that the group W contains the standard wreath product $B \wr \mathbb{Z}$, so always has exponential growth.

2.1. Imbedding in groups of subexponential growth. Our goal is, starting from a countable group H of locally subexponential growth, to construct a finitely generated group C of subexponential growth. We exhibit analogues of steps (i) and (ii) among *permutational wreath products*. Given groups H, G and an action of G on a set X , the *unrestricted permutational wreath product* is $H \wr_X G = H^X \rtimes G$, and the *restricted permutational wreath product* is the extension of finitely supported functions $X \rightarrow H$ by G .

Our previous work [4] gives a criterion, in terms of *inverted orbits*, that guarantees that the restricted permutational wreath product $W = H \wr_X G$ has subexponential growth as soon as H and G have subexponential growth. The inverted orbit of a point $x \in X$

under a word $w = g_1 \dots g_n$ in G is the set $\{xg_1 \dots g_n, xg_2 \dots g_n, \dots, xg_n, x\}$. If its cardinality may be bounded sublinearly in n , then W has subexponential growth. We compare subgroups $\langle G, f \rangle$ of the unrestricted wreath product with W to bound its growth.

Ad step (i), we show in Proposition 3.1 that for every group B there exists a group G that is a directed union of finite extensions of finite powers of B and such that $[G, G]$ contains B . In particular, if B has locally subexponential growth, so does G .

Ad step (ii), we need to consider separately the group P and the set X on which it acts. As a replacement for parallelogram-free sequences, we introduce *rectifiable* sequences, which are sequences (x_i) in X such that, for all $i \neq j$, there exists $g \in P$ with $x_i g = x_j$ and $x_k g \neq x_\ell$ for all $\ell \neq k \neq i$. We show that such sequences exist in the action of the first Grigorchuk group on the orbit of a ray, and more generally for all “weakly branched” groups.

The next step in the proof is an argument controlling the growth of a subgroup of the form $W = \langle G, f \rangle \leq B \wr_X G$, for a function $f: X \rightarrow B$ with sparse-enough (but infinite!) support. The rectifiability of the sequence (x_i) guarantees that functions with singleton support and arbitrary values in $[B, B]$ belong to W . Using the sparseness of the support of f , we show that balls in W can be approximated by balls in subgroups of restricted wreath products $\langle S \rangle \wr_X f$ for finite subsets S of B . By [4], these restricted wreath products have subexponential growth. We recall that, in general, a limit in Cayley topology of groups of subexponential growth may have exponential growth [6, Theorem C]; the Cayley topology on the space of finitely generated groups is the topology in which groups are close if their labeled Cayley graphs agree on a large ball. We control more precisely the approximation of W so that the growth estimates pass to the limit. Finally, in contrast with standard wreath products, the space X is not homogeneous, so an extra condition of stabilisation of balls around the x_i is required (even to ensure that W be amenable).

2.2. Distortion. We then apply this construction to produce, in §7, groups of subexponential growth all of whose imbeddings in Hilbert have arbitrarily large distortion in Hilbert space. We do this in three steps: first, given a sequence of finite groups H_1, H_2, \dots with bounded abelianisation and number of generators, we show that an arbitrary subset of the H_i ’s imbeds in a finitely generated group W of subexponential growth with controlled distortion; more precisely, the distortion constants of an imbedded H_i in W depends only on the previous H_j ’s, but not on H_i . Next, we assume that the H_i do not imbed uniformly in a class of metric spaces \mathcal{C} . Given any unbounded increasing function ρ , we select the subset of H_i ’s appropriately so that the distortion of W in any member of \mathcal{C} is worse than ρ . Finally, we use a construction of expanders H_i from [13] to exhibit, for every unbounded increasing function ρ , a group W with no ρ -distorted imbedding in uniformly convex Banach spaces of fixed uniformity modulus.

3. IMBEDDING IN THE DERIVED SUBGROUP

Let B be a group. A group G is *hyper- B* if it is a directed union of finite extensions of finite powers of B .

Proposition 3.1. *Let B be a group. Then there exists a hyper- B group G such that $[G, G]$ contains B as a subgroup. In particular, if B has locally subexponential growth, then so does G .*

If B is infinite, then G may furthermore be supposed to have the same cardinality as B .

We introduce the following notation. For groups H, U we denote by

$$H \wr^{\text{f.v.}} U = \{(\phi, u) \in H^U \times U : \#\phi(U) < \infty\}$$

the subgroup of the unrestricted wreath product $H^U \rtimes U$ in which the configurations take finitely many values. Note that it is a subgroup, because if $(\phi, u)^{-1}(\phi', u') = (\phi'', u^{-1}u')$ then $\phi''(U) \subseteq \phi(U)^{-1}\phi'(U)$ is finite.

Lemma 3.2. *If H is a hyper- B group and U is locally finite, then $H \wr^{\text{f.v.}} U$ is a hyper- B group.*

Proof. We first show that $H \wr^{\text{f.v.}} U$ is hyper- H . By hypothesis, U is a directed union of finite subgroups E . The partitions \mathcal{P}_0 of U into finitely many parts also form a directed poset; and for every such partition \mathcal{P}_0 and every finite $E \leq U$ there exists a finite partition \mathcal{P} of U that is invariant under E and refines \mathcal{P}_0 , namely the wedge (= least upper bound) of all E -images of \mathcal{P}_0 .

Consider now the directed poset of pairs (E, \mathcal{P}) consisting of finite subgroups $E \leq U$ and E -invariant partitions of U . Consider the corresponding subgroups $H^{\mathcal{P}} \rtimes E$ of $H \wr^{\text{f.v.}} U$. If $(E, \mathcal{P}) \leq (E', \mathcal{P}')$ then $H^{\mathcal{P}} \rtimes E$ is naturally contained in $H^{\mathcal{P}'} \rtimes E'$, so these subgroups of $H \wr^{\text{f.v.}} U$ form a directed poset, which exhausts $H \wr^{\text{f.v.}} U$.

It follows that $H \wr^{\text{f.v.}} U$ is a hyper- H group. Now a hyper-(hyper- B) group is itself hyper- B . \square

Lemma 3.3. *Let B be a group. Then there exists a subgroup C of B , containing $[B, B]$, such that B/C is torsion and $C/[B, B]$ is free abelian.*

Proof. $B/[B, B] \otimes_{\mathbb{Z}} \mathbb{Q}$ is a \mathbb{Q} -vector space, hence has a basis, call it X . It generates a free abelian group $\mathbb{Z}X$ within $B/[B, B]$, whose full preimage in B we call C . Then $B/C \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ so B/C is torsion. \square

We set up the following notation for the proof of Proposition 3.1. We choose a subgroup $C \leq B$ as in Lemma 3.3 and write $T := B/C$. We choose a basis X of $C/[B, B]$, for every $x \in X$ we choose an element $b_x \in C$ representing it, and we define a homomorphism $\theta_x: C \rightarrow \langle b_x \rangle$, trivial on $[B, B]$, by $\theta_x(b_x) = b_x^{-1}$ and $\theta_x(b_y) = 1$ for all $y \neq x \in X$. In particular, we have for all $b \in C$

$$b \cdot \prod_{x \in X} \theta_x(b) \in [B, B]$$

and the product is finite. Let F be a locally finite group of cardinality $> \#X$, and fix an imbedding of X in $F \setminus \{1\}$.

We write $\pi: B \rightarrow T$ the natural projection, and define a set-theoretic section $\sigma: T \rightarrow B$ as follows. Since T is torsion, it is locally finite, hence may be written as a union $T = \bigcup_{\alpha} T_{\alpha}$ of finite groups. Assume σ has already been defined on $T'_{\alpha} := \bigcup_{\beta < \alpha} T_{\beta}$. Choose a transversal T''_{α} of T'_{α} in T_{α} , namely a set of coset representatives of T'_{α} in T_{α} , and define σ arbitrarily on T''_{α} . Extend it then to T_{α} by $\sigma(t''t') = \sigma(t'')\sigma(t')$ for $t'' \in T''_{\alpha}, t' \in T'_{\alpha}$.

We consider first the group $G_0 = B \wr^{\text{f.v.}} (T \times F)$, and define a map $\Phi: B \rightarrow G_0$ as follows:

$$\Phi(b) = (\phi, \pi(b), 1) \text{ with } \phi(t, f) = \begin{cases} b & \text{if } f = 1, \\ \theta_f(\sigma(t)b\sigma(t\pi(b))^{-1}) & \text{if } f \in X, \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 3.4. *The map Φ_0 is well-defined and is an injective homomorphism into G_0 .*

Proof. To see that Φ_0 is well-defined, note that the argument $\sigma(t)b\sigma(t\pi(b))^{-1}$ belongs to $\ker(\pi) = C$, so that θ_f may be applied to it.

We next show that the image of Φ_0 belongs to G_0 . Consider $b \in B$. Let α be such that $\pi(b)$ belongs to the finite group T_{α} . Now given $t \in T$, write it using T_{α} and transversal elements as $t''_{\omega} \dots t''_{\beta} u$ with $\omega > \dots > \beta > \alpha$ and $u \in T_{\alpha}$. Then $\sigma(t) =$

$\sigma(t''_\omega) \dots \sigma(t''_\beta)\sigma(u)$ and $\sigma(t\pi(b)) = \sigma(t''_\omega) \dots \sigma(t''_\beta)\sigma(u\pi(b))$, so that $\sigma(t)b\sigma(t\pi(b))^{-1}$ is conjugate to $\sigma(u)b\sigma(u\pi(b))^{-1}$, and therefore $\theta_f(\sigma(t)b\sigma(t\pi(b))^{-1}) = \theta_f(\sigma(u)b\sigma(u\pi(b))^{-1})$ takes only finitely many values because θ_f vanishes on $[B, B]$. Also, $\theta_f(\sigma(t)b\sigma(t\pi(b))^{-1}) = 1$ except for finitely many values of $f \in X$. In summary, the function $\phi \in B^{T \times F}$ is such that $\phi(t, f)$ takes only finitely many values.

It is clear that Φ_0 is injective: if $b \in B \setminus C$ then its image in T is non-trivial, while if $b \in C \setminus \{1\}$ then $\phi(1, 1) = b$ is non-trivial. It is a homomorphism because all θ_f are homomorphisms. \square

Lemma 3.5. *We have $\Phi_0(C) \leq [G_0, G_0]$.*

Proof. If $b \in [B, B]$ then clearly $\Phi(b) \in [G_0, G_0]$. Since C is generated by $[B, B] \cup \{b_x\}_{x \in X}$, it suffices to consider $b = b_x$.

We define $g \in G_0$ by

$$g = (\phi, 1, 1) \text{ with } \phi(t, f) = \begin{cases} b_x & \text{if } t = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\Phi_0(b_x) = [(1, 1, x^{-1}), g] \in [G_0, G_0]$ as was to be shown. \square

We next define

$$G_1 = G_0 \wr^{\text{f.v.}} (\mathbb{Q}/\mathbb{Z})$$

and, for all $n \in \mathbb{N}$, a map $\Phi_{1,n}: B \rightarrow G_1$ by

$$\Phi_{1,n}(b) = (\phi, 1) \text{ with } \phi(r) = \begin{cases} \Phi_0(b) & \text{if } r \in [0, 1/n), \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 3.6. *For every $n \in \mathbb{N}$, the map $\Phi_{1,n}$ is an injective homomorphism, and $\Phi_{1,1}(B) \leq [G_1, G_1]$.*

Proof. The first assertion is clear, since Φ_0 is an injective homomorphism by Lemma 3.4. We know from Lemma 3.5 that $\Phi_{1,n}(C)$ is contained in $[G_1, G_1]$.

Consider now $b \in B$. Since B/C is torsion, there exists $n \in \mathbb{N}$ such that $b^n \in C$. We identify \mathbb{Q}/\mathbb{Z} with $[0, 1) \cap \mathbb{Q}$ and define $g \in G_1$ by

$$g = (\phi, 1) \text{ with } \phi(r) = \Phi_0(b)^{\lfloor rn \rfloor} \text{ for } r \in [0, 1) \cap \mathbb{Q}.$$

Then

$$\Phi_{1,1}(b) = [g, (1, 1/n)] \cdot \Phi_{1,n}(b^{-n}) \in [G_1, G_1]. \quad \square$$

Proof of Proposition 3.1. For the first assertion, we consider $G = G_1$ as above, with $\Phi_{1,1}: B \rightarrow G$ an injective homomorphism into $[G, G]$ by Lemma 3.6.

Assume that B has locally subexponential growth, and consider a finite subset S of G . Then there exists a subgroup of G that contains S and is virtually a finite power of B , hence has subexponential growth. This shows that G has locally subexponential growth.

Finally, we may replace G by a subgroup H with the same cardinality as B , if B is infinite: for each $b \in B$, choose a finite subset S_b of G such that $\Phi(b) \in [\langle S_b \rangle, \langle S_b \rangle]$, and a subgroup G_b , containing S_b , that is virtually a finite power of B . Consider the group H generated by the union of the G_b . As soon as B is infinite, all G_b have the same cardinality as B , and so does H . \square

4. ORBITS AND INVERTED ORBITS

Let $G = \langle S \rangle$ be a finitely generated group acting on the right on a set X . We consider X as the vertex set of a graph still denoted X , with for all $x \in X, s \in S$ an edge labelled s from x to xs . We denote by d the path metric on this graph.

Definition 4.1. A sequence (x_0, x_1, \dots) in X is *spreading* if for all R there exists N such that if $i, j \geq N$ and $i \neq j$ then $d(x_i, x_j) \geq R$.

Example 4.2. If all x_i lie in order on a geodesic ray starting from x_0 (for example if X itself is a ray starting from x_0) and for all i we have $d(x_0, x_{i+1}) \geq 2d(x_0, x_i)$, then (x_i) is spreading.

Lemma 4.3. Equivalently, a sequence (x_0, x_1, \dots) in X is spreading if and only if for all R there exists N such that if $i \neq j$ and $i \geq N$ then $d(x_i, x_j) \geq R$.

Proof. Assume the converse, namely $d(x_i, x_j) < R$ along a sequence with $i \rightarrow \infty$ and $j \nrightarrow \infty$. Then, up to passing to a subsequence, j may be assumed constant. There are then $i, i' \rightarrow \infty$ with $i \neq i'$ and $d(x_i, x_{i'}) < 2R$, so (x_i) is not spreading. \square

Definition 4.4. A sequence (x_i) in X *locally stabilises* if for all R there exists N such that if $i, j \geq N$ then the S -labelled radius- R balls centered at x_i and x_j in X are equal.

Definition 4.5. A sequence of points (x_i) in X is *rectifiable* if for all $i \neq j$ there exists $g \in G$ with $x_i g = x_j$ and $x_k g \neq x_\ell$ for all $k \notin \{i, \ell\}$.

It is clear that, if G is finitely generated and X is infinite, then it admits spreading and locally stabilizing sequences. Also, a subsequence of a spreading or locally stabilizing sequence is again spreading, respectively locally stabilizing. We give below a general construction of rectifiable sequences, and later a concrete example in the first Grigorchuk group.

4.1. Rectifiable sequences. Consider a group G acting on a set X . We recall that the *fixator* of the subset $Y \subseteq X$ is the set $\text{Fix}(Y) := \{g \in G : yg = y \text{ for all } y \in Y\}$.

Definition 4.6 (Abért [1]). The group G *separates* X if for every finite subset $Y \subseteq X$ and every $y_0 \notin Y$ there exists $g \in \text{Fix}(Y)$ with $y_0 g \neq y_0$.

Lemma 4.7. Let G be a group acting on a non-empty set X and separating it. Then there exists a rectifiable sequence (x_i) in X .

Proof. First recall that if G separates X , then X is infinite, and, moreover, for any finite set $Y \subset X$ and any $y_0 \notin Y$ the orbit of y_0 under the fixator of Y is finite. Indeed, if this orbit \mathcal{O} were finite, the fixator of $Y' = Y \cup \mathcal{O} \setminus \{y_0\}$ would not be able to move $y_0 \notin Y'$.

We start by choosing an arbitrary $x_0 \in X$. We construct iteratively a sequence (x_i) in X and a sequence (g_i) in G . For all $i \geq 1$, we choose g_i in the fixator of $\{x_0 g_j^{\pm 1} g_k^{\pm 1} g_\ell^{\pm 1} : 1 \leq j, k, \ell < i\} \setminus \{x_0\}$ that moves x_0 , and we write $x_i := x_0 g_i^{-1}$.

Now given $i \neq j$, we consider $g = g_i g_j^{-1}$ and note that we have $x_i g = x_0 g_j^{-1} = x_j$. If we additionally had $x_k g = x_\ell$, then we would have $x_0 g_k^{-1} g_i g_j^{-1} g_\ell = x_0$. In this equality, if the largest index among $\{i, j, k, \ell\}$ is k , then this contradicts the choice of g_k , since $x_0 g_k^{-1}$ is not in the ball of radius 3 around x_0 . If the largest index is ℓ , the same argument holds by symmetry. If the largest index is i , then we get $x_0 g_k^{-1} g_j^{-1} g_\ell = x_0$, and this again contradicts the choice of the group element with next-largest index. Finally, all cases in which the largest index occurs twice or more reduce to $i = j, i = k$ or $k = \ell$. \square

We quote from Abért ([1, Proof of Corollary 1.4], see also [6, Lemma 6.11]) that the action of weakly branch groups separates the boundary of their tree. Since the first Grigorchuk group is weakly branched (see [9, Theorem 1] or [3, Proposition 1.25]), it provides

by Lemma 4.7 an example of a group action with rectifiable sequences. We also see it directly in the following §.

4.2. An orbit for the first Grigorchuk group. In this subsection, we consider the first Grigorchuk group $G_{012} = \langle a, b, c, d \rangle$. Recall that it acts on set of infinite sequences $\{0, 1\}^\infty$ over a two-letter alphabet, which is naturally the boundary of a binary rooted tree. We denote by $X = 1^\infty G_{012}$ the orbit of the rightmost ray. We construct explicitly a spreading, locally stabilizing, rectifiable sequence for the action of G_{012} on X . For all $n \in \mathbb{N}$, we define $x_n = 0^n 1^\infty$.

Lemma 4.8. *For all $m, n \in \mathbb{N}$,*

- (1) *the marked balls of radius $2^{\min(m, n)}$ in X around x_n and x_m coincide;*
- (2) *the distance $d(x_m, x_n)$ is $|2^m - 2^n|$;*
- (3) *there exists $g_{m, n} \in G_{012}$ of length $|2^m - 2^n|$ with $x_m g_{m, n} = x_n$ and $x_k g_{m, n} \neq x_\ell$ for all $(k, \ell) \neq (m, n)$;*

Proof. (1,2) Consider the map $\sigma : a \mapsto c, b \mapsto d^a, c \mapsto b^a, d \mapsto c^a$. It defines a self-map of X by sending $1^\infty g$ to $1^\infty \sigma(g)$. A direct calculation shows that it sends $x \in X$ to $0x$.

Since σ is 2-Lipschitz on words of even length in $\{a, b, c, d\}$, it maps the ball of radius n around x to the ball of radius $2n$ around $0x$. Its image is in fact a net in the ball of radius $2n$: two points at distance 1 in the ball of radius n around x will be mapped to points at distance 1 or 3 in the image, connected either by a path a or by a segment $a-b-a, a-c-a$ or $a-d-a$. In particular, the 2^n -neighbourhoods of the balls about the x_m coincide for all $m \geq n$.

The geometric image of the Schreier graph X is that of a ray. The point x_n is at position 2^n along this ray.

(3) Note, first, that there exists $g_{i, j}$ with $x_i g_{i, j} = x_j$, because the rays ending in 1^∞ form a single orbit. Note, also, that we have $x_k g_{i, j} = x_\ell$ for either finitely many $(k, \ell) \neq (i, j)$ or for all but finitely many (k, ℓ) , because there is a level N at which the decomposition of $g_{i, j}$ consists entirely of generators; if the entry at 0^N of $g_{i, j}$ is trivial or ‘ d ’ then all but finitely many of the x_k are fixed; while otherwise (up to increasing N by at most one) we may assume it is an ‘ a ’; then $0^{N+1} g_{i, j} = 0^N 1$, so $x_k \neq x_\ell$ for all $k > N + 1$.

If the entry at 0^N of $g_{i, j}$ is trivial, then we multiply $g_{i, j}$ with the element $(ab)^2$ in the rigid stabilizer of 0^M for some $M > \max(N, i)$, so as to fall back to the second case.

Then, for each pair $(k, \ell) \neq (i, j)$ with $x_k g_{i, j} = x_\ell$, we multiply $g_{i, j}$ with the element $(ab)^2$ in the rigid stabilizer of $0^\ell 1$, so as to destroy the relation $x_k g_{i, j} = x_\ell$.

The resulting element $g_{i, j}$ satisfies the required conditions. \square

5. SUBEXPONENTIAL GROWTH OF WREATH PRODUCTS

In this section, we show how some permutational wreath products have subexponential growth.

Definition 5.1. The group G acting on X has the *subexponential wreathing property* if for any finitely generated group of subexponential growth H the restricted wreath product $H \wr_X G$ has subexponential growth.

Lemma 5.2. *Let f be a positive sublinear function, namely $f(n)/n \rightarrow 0$ as $n \rightarrow \infty$. Then f is bounded from above by a concave sublinear function.*

Proof. For every $\theta \in (0, 1)$, let n_θ be such that $f(n) - \theta n$ is maximal. If $n \in [n_\theta, n_\zeta]$ for some $\zeta < \theta$, and $[\zeta, \theta]$ is a minimal interval with this property, then define $\bar{f}(n)$ on $[n_\theta, n_\zeta]$ by linear interpolation between $(n_\theta, f(n_\theta))$ and $(n_\zeta, f(n_\zeta))$. Clearly $\bar{f} \geq f$, and $\bar{f}(n)/n$ is decreasing and coincides infinitely often with $f(n)/n$, so it converges to 0. \square

Lemma 5.3. *Let the Schreier graph of X have linear growth, and assume that G has sublinear inverted orbit growth on X . Assume also that G has subexponential growth. Then G has the subexponential wreathing property.*

Proof. We essentially follow [4, Lemma 5.1].

Fix some $x_0 \in X$ and let $\rho(n)$ be the growth of inverted orbits starting from x_0 . By assumption, $\rho(n)/n \rightarrow 0$, and there is a constant C such that the ball of radius n around x_0 has cardinality $\leq Cn$.

Let H be a group of subexponential growth, and choose a finite generating set for H . By Lemma 5.2, there exists a log-concave subexponential function \bar{v}_H bounding the growth function $v_H(n)$ of H .

We view $H \wr_X G$ as generated by the generating set of G and the imbedding of the generating set of H as functions supported at $\{x_0\}$.

Consider an element $(c, g) \in H \wr_X G$ of norm R . The function $c: X \rightarrow H$ has support of cardinality at most $\rho(R)$, and this support is contained in the ball of radius R around x_0 . The values of c belong to H , and their total norm is $\leq R$. Therefore, the cardinality of the ball of radius R in $H \wr_X G$ is bounded from above as

$$v_{H \wr_X G}(R) \leq v_G(R) \left(\frac{CR}{\rho(R)} \right) \bar{v}_H \left(\frac{R}{\rho(R)} \right)^{\rho(R)}.$$

Since it is a product of subexponential functions, it is itself subexponential. \square

We now quote [4, Proposition 4.4]: the inverted orbit growth of the first Grigorchuk group G_{012} on $X = \mathbf{1}^\infty G_{012}$ is sublinear (actually of the form n^α for some $\alpha < 1$); therefore, by Lemma 5.3, the action of G_{012} on X has the subexponential wreathing property. (It follows from [5] that all Grigorchuk groups G_ω also have the subexponential wreathing property, as soon as $\omega \in \{0, 1, 2\}^\infty$ contains infinitely many copies of each symbol.)

6. THE CONSTRUCTION OF W

Using the results of the previous section, we select a group G acting on a set X , and a separating, spreading, locally stabilizing sequence (x_i) of elements of X .

Let (b_1, b_2, \dots) be a sequence in B . We will construct a rapidly increasing sequence $0 \leq n(1) < n(2) < \dots$ later; assuming this sequence given, we define $f: X \rightarrow B$ by

$$f(x_{n(1)}) = b_1, \quad f(x_{n(2)}) = b_2, \quad \dots, \quad f(x) = 1 \text{ for other } x.$$

We then consider the subgroup $W = \langle G, f \rangle$ of the unrestricted wreath product $B^X \rtimes G$.

Lemma 6.1. *Denote by B_0 the subgroup of B generated by $\{b_1, b_2, \dots\}$. If the sequence (x_i) is separating, then W contains $[B_0, B_0]$ as a subgroup.*

Proof. Without loss of generality and to lighten notation, we rename B_0 into B . We also denote by $\iota: B \rightarrow B^X \rtimes G$ the imbedding of B mapping the element $b \in B$ to the function $X \rightarrow B$ with value b at x_0 and 1 elsewhere. We shall show that W contains $\iota([B, B])$. For this, denote by H the subgroup $\iota(B) \cap W$.

We first consider an elementary commutator $g = [b_i, b_j]$. Let $g_i, g_j \in G$ respectively map x_i, x_j to x_0 , and be such that $g_i g_j^{-1}$ maps no x_k to x_ℓ with $k \neq \ell$, except for $x_i g_i g_j^{-1} = x_j$. Consider $[f^{g_i}, f^{g_j}] \in W$; it belongs to B^X , and has value $[b_i, b_j]$ at x_0 and is trivial elsewhere, so equals $\iota(g)$ and therefore $\iota(g) \in H$.

We next show that H is normal in B^X . For this, consider $h \in H$. It suffices to show that $h^{\iota(b_i)}$ belongs to H for all i . Now $h^{\iota(b_i)} = h^{f^{g_i}}$ belongs to H , and we are done. \square

Proposition 6.2. *Let G be a group acting on X . Let the sequence (x_i) in X be spreading and locally stabilizing. Let a sequence of elements (b_i) be given in the group B , all of the same order $\in \mathbb{N} \cup \{\infty\}$.*

For all $i \in \mathbb{N}$, let f_i be the finitely supported function $X \rightarrow B$ with $f_i(x_{n(j)}) = b_j$ for all $j \leq i$, all other values being trivial, and denote by W_i the group $\langle f_i, G \rangle$.

Then for every increasing sequence $(m(i))$ there is a choice of $(n(i))$ such that the ball of radius $m(i)$ in W coincides with the ball of radius $m(i)$ in W_i , via the identification $f \leftrightarrow f_i$.

Furthermore, the term $n(i)$ depends only on $m(i)$ and on the ball of radius $m(i)$ in $\langle b_1, \dots, b_{i-1} \rangle$.

Proof. Choose $n(i)$ such that $d(x_j, x_k) \geq m(i)$ for all $j \neq k$ with $k \geq n(i)$, and such that the balls of radius $m(i)$ around $x_{n(i)}$ and x_j coincide for all $j > n(i)$.

Consider then an element $h \in W$ in the ball of radius $m(i)$, and write it in the form $h = (c, g)$ with $c: X \rightarrow B$ and $g \in G$. The function c is a product of conjugates of f by words of length $< R$. Its support is therefore contained in the union of balls of radius $m(i) - 1$ around the x_j , with j either $\geq n(i)$ or of the form $n(k)$ for $k < i$. In particular, the entries of c are in $\langle b_1, \dots, b_{i-1} \rangle \cup \bigcup_{j \geq i} \langle b_j \rangle$. For $j > n(i)$, the restriction of c to the ball around x_j is determined by the restriction of c to the ball around $x_{n(i)}$, via the identification $b_i \mapsto b_j$, because the neighbourhoods in X coincide and all cyclic groups $\langle b_j \rangle$ are isomorphic.

It follows that the element $h \in W$ is uniquely determined by the corresponding element in W_i . \square

Corollary 6.3. *Let G be a group acting on X with the subexponential wreathing property. Let the sequence (x_i) be spreading and locally stabilizing. Assume that B has locally subexponential growth, and that the sequence $(n(i))$ grows sufficiently fast.*

Then W has subexponential growth.

Proof. Let $Z = \langle z \rangle$ be a cyclic group whose order (possibly ∞) is divisible by the order of the b_i 's. We replace B by $B \times Z$ and each b_i by $b_i z$, so as to guarantee that all generators in B have the same order.

Let ϵ_i be a decreasing sequence tending to 1. Denote by v_i the growth function of the group W_i introduced in Proposition 6.2, and by w the growth function of W . Let $m(i)$ be such that

$$v_i(m(i)) \leq \epsilon_i^{m(i)}.$$

Such an $m(i)$ exists, because $B \wr_X G$ has locally subexponential growth. Since the balls of radius $m(i)$ coincide in W and W_i , we also have $w(m(i)) \leq \epsilon_i^{m(i)}$. Then, if $R > m(i)$, we get

$$w(R) \leq \epsilon_i^{R+m(i)},$$

so $\limsup_{R \rightarrow \infty} \sqrt[R]{w(R)} \leq \epsilon_i$. Since this holds for all i , the growth of W is subexponential. \square

Proof of Theorem A. By Proposition 3.1, the countable, locally subexponentially growing group B imbeds in $[H, H]$ for a countable, locally subexponentially growing group H . By Lemma 6.1, $[H, H]$ imbeds in W , and by Corollary 6.3, the finitely generated group W has subexponential growth. \square

Remark 6.4. If the sequence (x_i) is only spreading, or only stabilizing, then it may happen that W has subexponential growth, even if the sequence $(n(i))$ grows arbitrarily fast.

Proof. We first consider an example where the sequence (x_i) is spreading but not stabilizing. Consider $G = G_{012}$ acting on $X = \mathbf{1}^\infty G$, and let P denote the stabilizer of $\mathbf{1}^\infty$ so that $X = P \backslash G$. Since the action is faithful, we have $\bigcap_{g \in G} P^g = 1$, and in fact $\bigcap_{g \in T} P^g = 1$ for a sequence T in G such that $(\mathbf{1}^\infty t: t \in T)$ is spreading. Take $B = \langle z \rangle \cong \mathbb{Z}$ and define $f: X \rightarrow B$ by $f(\mathbf{1}^\infty t) = z$ for all $t \in T$, all other values being 1. Then $\langle G, f \rangle \cong \mathbb{Z} \wr G$ has exponential growth.

We next consider an example where the sequence (x_i) is stabilizing but not spreading. Again, consider $G = G_{012}$ acting on X , and consider a spreading, stabilizing sequence (x_{2i}) in X . Set $x_{2i-1} = x_i a$. Consider $B = G_{012}$, and note that, since G does not satisfy any law, there are sequences $(g_1, h_1), (g_2, h_2), \dots$ of pairs of elements of G such that the groups $\langle g_i, h_i \rangle$ converge to a free group of rank 2 in the Cayley topology. Set then $f(x_{2i}) = h_i$ and $f(x_{2i-1}) = g_i$, and note that $\langle G, f \rangle$ contains a free group. \square

7. DISTORTION

We modify slightly the construction so as to produce groups with particularly bad distortion in their imbeddings in metric spaces.

Let $H = \langle T \rangle$ be a group with fixed generating set. We denote by $\| \cdot \|_T$ the word norm on H , and consider another norm on $[H, H]$. For this, let us say that a word w in the free group F_T is *balanced* if it belongs to $[F_T, F_t]$; namely, if it contains as many t 's as t^{-1} 's for every letter $t \in T$. The *perfect norm* on $[H, H]$ is

$$\|g\|_{\text{perfect}} = \min\{\|w\| : w \in F_T \text{ is a perfect word representing } g\}.$$

We denote by $d_T(x, y) = \|xy^{-1}\|_T$ and $d_{\text{perfect}}(x, y) = \|xy^{-1}\|_{\text{perfect}}$ the corresponding distances.

We shall be interested in sequences $(H_i)_{i \in \mathbb{N}}$ of finite groups satisfying the following properties:

- (H1) Each group H_i is d -generated, and a symmetric generating set T_i of cardinality d has been fixed;
- (H2) The abelianisation of H_i has bounded cardinality;
- (H3) The inclusion map $([H_i, H_i], d_{\text{perfect}}) \rightarrow (H_i, d_{T_i})$ is $(J^{-1}, 1)$ -bi-Lipschitz for a constant J independent of i .

The heart of the argument is the following variant of Corollary 6.3:

Proposition 7.1. *Let $(H_i)_{i \in \mathbb{N}}$ be a sequence of finite groups satisfying (H1-H3). Then there exists a family of groups $(W_S)_{S \subset \mathbb{N}}$ indexed by subsets S of \mathbb{N} , each of subexponential growth, with the following property: for all $s \in S$, there is an imbedding $\Psi_s : [H_s, H_s] \rightarrow W_S$ that is (K, L) -bi-Lipschitz, and the constants K, L depend only on $\{H_i : i < s\}$.*

Furthermore, if $S = \{s(1), s(2), \dots\}$, then $W_{\{s(1), \dots, s(i+1)\}}$ is constructed out of $W_{\{s(1), \dots, s(i)\}}$, and the sequence $(W_{\{s(1), \dots, s(i)\}})_{i \in \mathbb{N}}$ converges to W_S in the Cayley topology.

Proof. Up to replacing $(H_i)_{i \in \mathbb{N}}$ by $(H_i)_{i \in S}$, we lighten notation and suppose $S = \mathbb{N}$ or a prefix $\{1, 2, \dots, n\}$ thereof.

Let us write $T_i = \{t_{i,1}, \dots, t_{i,d}\}$ the generating set of H_i . We consider the restricted direct product $B = \prod_{i \geq 1} H_i$. It is a countable, locally finite group, generated by $\{t_{1,1}, \dots, t_{1,d}, t_{2,1}, \dots\} = \{b_1, b_2, \dots\}$. We consider the group W constructed in Section 6, noting that the sequence $n(i)$ may be chosen, by Corollary 6.3, such that W has subexponential growth, and that $n(i)$ depends only on $H_1, H_2, \dots, H_{\lfloor i/d \rfloor - 1}$.

Consider $[H_i, H_i]$ as a subgroup of W , imbedded as the functions $X \rightarrow H_i \subset B$ supported only at $x_{n(di)}$. This is an imbedding by Lemma 6.1. Denote by $\Psi_i : H_i \rightarrow W$ this imbedding.

Assume that $n(1), \dots, n(di)$ have already been chosen; and note that their choice relies only on H_1, \dots, H_{i-1} . Recall also that $f(x_{n(di-d+j)}) = t_{i,j}$ in the construction of W . We now show that there exist constants K, L independent of H_i such that the imbedding $\Psi_i : [H_i, H_i] \rightarrow W$ is (K, L) -bi-Lipschitz. In other words, independently of i , the distortion of $[H_i, H_i]$ in W is at worst $\rho(t) = tK/L$.

Let $g_1, \dots, g_d \in G$ be such that $x_{n(di-d+j)}g_j = x_{n(di)}$ and the only x_k mapped to another x_ℓ under $g_j'g_j^{-1}$ are $x_{n(di-d+j')}g_j'g_j^{-1} = x_{n(di-d+j)}$; such elements exist because

(x_i) is separating. Let L' be an upper bound for the lengths of all g_1, \dots, g_d . This condition ensures that the functions f^{g_1}, \dots, f^{g_d} have disjoint support except at $x_{n(di)}$ or where they coincide.

Here is an explicit way of computing the imbedding $\Psi_i: [H_i, H_i] \rightarrow W$: for $h \in [H_i, H_i]$, write it as a minimal-length balanced word in T_i , and map each letter $t_{i,j}$ to f^{g_j} .

On the one hand, $\|\Psi_i(h)\|_W \leq (2L' + 1)\|h\|_{\text{perfect}}$ because each letter gets mapped to a word of length $1 + 2\|g_j\| \leq 1 + 2L'$; on the other hand, $\|\Psi_i(h)\|_W \geq \|h\|$, because at most one element of T is contributed by each generator of W . Combining with the uniformly bi-Lipschitz map $([H_i, H_i], d_{\text{perfect}}) \rightarrow (H_i, d)$ gives the result. \square

7.1. Imbeddings. Let \mathcal{C} be a class of metric spaces. Given a sequence of metric spaces such as $((H_i, d))_{i \in \mathbb{N}}$, we say that it *does not imbed uniformly* in \mathcal{C} if the following holds: there exists a constant M such that, if $\mathcal{X} \in \mathcal{C}$ and $(\Phi_i: H_i \rightarrow \mathcal{X})$ is a sequence of 1-Lipschitz imbeddings then, for all $t \in \mathbb{R}$, there are $i \in \mathbb{N}$ and $x, y \in H_i$ with $d(x, y) \geq t$ and $d(\Phi_i(x), \Phi_i(y)) \leq M$.

Corollary 7.2. *Let $(H_i)_{i \in \mathbb{N}}$ be a sequence of finite groups satisfying (H1-H3). Assume furthermore that (H_i) does not imbed uniformly in a class \mathcal{C} of metric spaces. Let ρ be any unbounded increasing function $\mathbb{R}_+ \rightarrow \mathbb{R}_+$. Then there exists a finitely generated group W of subexponential growth such that every imbedding of W in an element of \mathcal{C} has worse distortion than ρ .*

Proof. Since the groups H_i have uniformly bounded abelianisation, there exists a (possibly larger) constant M such that, if $\mathcal{X} \in \mathcal{C}$ and $(\Phi_i: H_i \rightarrow \mathcal{X})$ is a sequence of 1-Lipschitz imbeddings then, for all $t \in \mathbb{R}$, there are $i \in \mathbb{N}$ and $x, y \in [H_i, H_i]$ with $d(x, y) \geq t$ and $d(\Phi_i(x), \Phi_i(y)) \leq M$.

The group W will be of the form $W = W_S \times W_{S'}$, for sequences $S = \{s(1), s(3), s(5), \dots\}$ and $S' = \{s(2), s(4), \dots\}$ that we construct iteratively. Assume that $s(1), \dots, s(i-1)$ have been constructed. The terms $s = s(i)$ and $s' = s(i+1)$ have not yet been determined, but we already know the constants $K_i, L_i, K_{i+1}, L_{i+1}$ such that the imbedding of $[H_s, H_s]$ into W_S or $W_{S'}$ will be (K_i, L_i) -bi-Lipschitz and the imbedding of $[H_{s'}, H_{s'}]$ into $W_{S'}$ or W_S will be (K_{i+1}, L_{i+1}) -bi-Lipschitz.

We now make use of the unbounded function ρ . Let $t_i \in \mathbb{R}$ be large enough so that $\rho(t_i) > L_{i+1}M$. Since the (H_i) do not imbed uniformly, we can choose s large enough so that there exist $x, y \in [H_s, H_s]$ with $d(x, y) \geq t_i/K_i$ and $d(\Phi_s(x), \Phi_s(y)) \leq M$ in any 1-Lipschitz imbedding Φ_s of H_s into an element of \mathcal{C} . Without loss of generality, the sequences (t_i) and $(s(i))$ are strictly increasing. This determines $s = s(i)$, and finishes the inductive construction of S and S' .

Let us now check that the group W just constructed has the desired property. Let $\mathcal{X} \in \mathcal{C}$ be a metric space in the specified class, and let $\Phi: W \rightarrow \mathcal{X}$ be a 1-Lipschitz imbedding. By composing with the imbedding of H_s in W_S or $W_{S'}$, we get for all $s \in S \cup S'$ imbeddings $\Phi_s = \Phi \circ \Psi_s$ of $[H_s, H_s]$ into \mathcal{X} .

Consider $t \in \mathbb{R}_+$, and suppose $t \geq t_1$. Let i be such that $t_{i-1} \leq t < t_i$. Set $s = s(i)$. Following the construction above, the imbedding Φ_s is (K_i, L_i) -Lipschitz, so there are $x, y \in [H_s, H_s]$ with $d(x, y) \geq t_i/K_i$ so $d(\Psi_s(x), \Psi_s(y)) \geq t_i$ while $d(\Phi_s(x), \Phi_s(y)) \leq L_i M$. This proves that the distortion ρ_Φ of Φ satisfies

$$\rho_\Phi(t) \leq \rho_\Phi(t_i) \leq L_i M < \rho(t_{i-1}) \leq \rho(t),$$

so the distortion of W is worse than ρ . \square

We note from the proof that the distortion of a single copy W_S is worse than ρ along an unbounded sequence.

7.2. Superexpanders. We now exhibit a sequence (H_i) with particularly bad imbedding properties. We recall the following definition from [13].

Consider $\alpha > 0$ and q a prime power. Denote by \mathbb{F}_q the elementary abelian group with q elements. Lafforgue denotes by $\mathcal{E}^{q,\alpha}$ the class of Banach spaces $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ such that for all $(x_i)_{i \in \mathbb{F}_q}$ in \mathcal{X} , writing $\widehat{\mathbb{F}_q}$ the group of characters of \mathbb{F}_q , one has

$$q^{-1} \sum_{\chi \in \widehat{\mathbb{F}_q}} \left\| q^{-1} \sum_{i \in \mathbb{F}_q} \chi(i) x_i \right\|_{\mathcal{X}}^2 \leq e^{-\alpha} \left(q^{-1} \sum_{i \in \mathbb{F}_q} \|x_i\|_{\mathcal{X}}^2 \right).$$

He notes that Hilbert space belongs to $\mathcal{E}^{q, \log q}$, and that every uniformly convex Banach space belongs to $\mathcal{E}^{q,\alpha}$ with α depending only on q and the uniform convexity modulus. He then gives the following construction of expanders:

Proposition 7.3 ([13, Théorème 5.1]). *There exists a sequence $(H_i)_{i \in \mathbb{N}}$ of finite quotients of a finitely generated group H such that the sequence of Cayley graphs of the H_i does not admit any uniform imbedding into a Banach space belonging to $\mathcal{E}^{q,\alpha}$ for some $\alpha > 0$ and q a prime power.*

Here is a concrete example: consider $H = \mathbf{SL}_3(\mathbb{F}_p[t])$ and its images H_i in $\mathbf{SL}_3(\mathbb{F}_p[t]/(t^i))$. In this situation, we have a few extra, useful properties, which we quote as a

Lemma 7.4. *Additionally, H may be supposed to be perfect.*

Proof. Since $\mathbb{F}_p[t]$ is a Euclidean domain, H is generated by elementary matrices. Furthermore, the classical identities $X_{i,j}(P+Q) = X_{i,j}(P)X_{i,j}(Q)$ and $X_{i,j}(PQ) = [X_{i,k}(P), X_{k,j}(Q)]$ between elementary matrices, when $\{i, j, k\} = \{1, 2, 3\}$, imply that H is generated by $A = \mathbf{SL}_3(\mathbb{F}_p)$ and $B = \langle X_{1,2}(t) \rangle$. Since A is perfect and $B^A = \{1 + tM : M \in M_3(\mathbb{F}_p) \text{ and } \text{tr}(M) = 0\}$ has no A -invariant element, H is also perfect. \square

We fix as generating set $T = A \cup B$, and denote by T_i its natural image in H_i .

Lemma 7.5. *Let H be a finitely generated group with finite abelianisation, and let $(H_i)_{i \in \mathbb{N}}$ be a family of quotients of H . Then the groups H_i satisfy all the hypotheses (H1-H3) stated at the beginning of §7.*

Proof. The abelianisations of the H_i are quotients of the abelianisation of H , and therefore have bounded cardinality.

If $H = \langle T \rangle$ is d -generated, then the groups H_i are naturally d -generated by the images T_i of T .

Let finally T' be a Schreier generating set of $[H, H]$: choose a transversal Ξ of $[H, H]$ in H , and set $T' = \Xi T \Xi^{-1} \cap [H, H]$. Note that T' is finite, since Ξ and T are finite. Represent each $t \in T'$ as a balanced word, and let J be the maximal length of these balanced words. Then

$$\|g\|_{T_i} \leq \|g\|_{\text{perfect}} \leq J \|g\|_{T'} \leq J \|g\|_{T_i} \text{ for all } g \in [H_i, H_i],$$

so the inclusion $([H_i, H_i], d_{\text{perfect}}) \rightarrow (H_i, d_{T_i})$ is $(J^{-1}, 1)$ -bi-Lipschitz. \square

Corollary 7.6. *Consider $\alpha > 0$, and let ρ be any unbounded increasing function $\mathbb{R}_+ \rightarrow \mathbb{R}_+$. Then there exists a finitely generated group W of subexponential growth such that every imbedding of W in a Banach space $\mathcal{X} \in \mathcal{E}^{q,\alpha}$, for q a prime power, has distortion worse than ρ_0 .*

In particular, let ρ be any unbounded increasing function $\mathbb{R}_+ \rightarrow \mathbb{R}_+$. Then there exists a finitely generated group W of subexponential growth such that every imbedding of W in Hilbert space has distortion worse than ρ_0 . \square

Proof. Consider the sequence of superexpanders $(H_i)_{i \in \mathbb{N}}$ given by Proposition 7.3; assume, thanks to Lemma 7.5, that it satisfies the hypotheses (H1-H3) stated at the beginning of §7.

By the proof of [13, Proposition 5.2], there exists a constant M , depending only on α , such that if $(\Phi_i: H_i \rightarrow \mathcal{X})$ is a sequence of 1-Lipschitz imbeddings into a Banach space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}}) \in \mathcal{E}^{q,\alpha}$ then for every $t \in \mathbb{R}$ and for all j large enough (depending on t) there exist $x, y \in H_j$ with $d(x, y) \geq t$ and $d_{\mathcal{X}}(\Phi_j(x), \Phi_j(y)) \leq M$.

Corollary 7.2 then applies. \square

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